Consistent Rational-Function Approximation for Unsteady Aerodynamics

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An improved method is developed for the approximation of generalized, unsteady aerodynamic forces by a rational transfer function in the Laplace domain. Whereas the previous methods produce an ill-conditioned eigenvalue problem when the optimized values of two or more poles of the transfer function are close to one another, the present scheme accounts for such frequent cases consistently. Also, the new method results in a large reduction in the computational cost of an optimized aerodynamic rational approximation when compared with the previous procedures for a given accuracy. These improvements are due to the use of higher order poles (as against the simple poles of conventional methods), without increasing the total number of aerodynamic states of the system, and they make the method applicable to routine transient response calculations. The method employs a nongradient optimizing process for the selection of the nonlinear parameters of the transfer function. Approximations are presented for the three-dimensional, subsonic aerodynamics of a high aspect ratio wing.

Introduction

THE representation of motion of an elastic structure under the influence of unsteady aerodynamic loads is described in terms of generalized coordinates $\xi(t)$ by the following matrix equation:

$$[M]\{\ddot{\xi}(t)\}+[C]\{\dot{\xi}(t)\}+[K]\{\xi(t)\}=\{Q(t)\} \tag{1}$$

where $[M]$, $[K]$, and $[C]$ are the generalized mass, stiffness, and damping matrices, respectively, and $\{Q(t)\}$ the generalized air-force vector in the time domain. The order of each generalized matrix is equal to the number of retained modes. Prior to the solution of matrix Eq. (1), it is required that the generalized air-force vector be available in the time domain. This is accomplished most easily if the Laplace transform representation of Eq. (1) is used. This yields

$$([M]s^2+[C]s+[K])\{\xi(s)\}=[Q(s)]\{\xi(s)\} \tag{2}$$

Equation (2) is said to represent a finite-state aeroelastic system that can be converted into a linear, time-invariant, state-space form for performing a stability analysis or control system design if each term of the unsteady aerodynamic matrix $[Q(s)]$ can be represented by a ratio of polynomials in s. Generally, the air forces can be determined only for pure oscillatory motion of a structure such as a lifting surface. However, since $[Q(s)]$ is analytic for a causal, stable, and linear system, it can be directly deduced from $[Q(i\omega)]$, which is obtained from an oscillatory theory. This is realized by approximating each term of the generalized air-force matrix $[Q(i\omega)]$ by a rational polynomial in $(i\omega)$ and then solving for the coefficients of the polynomial, which gives the least-squared-error fit with tabulated oscillatory air forces at given values of frequency. The transfer function matrix $[Q(s)]$ is then obtained by the replacement $s = i\omega$. In practice, it is more convenient to use the nondimensional reduced frequency $k = \omega b/U$ because the oscillatory aerodynamic data are generally available for the reduced frequencies. When this is done, the Laplace variable also becomes nondimensionalized such that $sb/U = ik$. There have been many approaches to this direct conversion process.[1-17] This basic method employed here is that of Roger[7] and Abel[8] who formulated a rational-function approximation for three-dimensional, subsonic aerodynamics by using a series of poles to represent the aerodynamic lags attributable to the wake. The poles are chosen to be the same for all elements of the transfer matrix. The success of the fit is dependent on the choice of the poles, which, in turn, is based on experience. This method is known as the conventional least-squares method because the parameters in the curve fit are determined by a least-squares technique. Tiffany and Adams[9] used a nongradient, nonlinear optimizer to select the values of lag parameters in the least-squares formulation, which gave the minimum total least-squared fit error with oscillatory data. They reported significant improvement and called the resulting method extended least-squares. Another approach, similar in many ways to the rational-function approximation of Roger[7] and Abel,[8] is that of Dowell.[10] He used a series of decaying exponentials in the time domain, which in the Laplace domain is represented by a series of simple poles. Recently, Peterson and Crawley[11] have published results employing a Newton-Raphson gradient optimization process to solve for the lag parameters of Dowell's exponential series in the time domain. Their results appear to have nonunique minima of the objective function, and they include the numerator and denominator coefficients of the transfer function as the free parameters for optimization. The latter feature is expected to make the optimization process less efficient, and the former results in a loss of accuracy and an erratic behavior of the objective function with the number of lag states.

The chief disadvantage of the optimized rational-function approximations has been their large incremental cost of optimization, which may limit their routine use in the transient response analysis. For example, the extended least-squares method of Tiffany and Adams[9] is about 20 times as expensive as an unoptimized method. The minimum-state method of Karpel[17] reduces the total number of augmented states in the aeroelastic model for a given accuracy but requires a computation time nearly 1000 times greater than that of an unoptimized method.[20] The present investigation leads to a form that enhances the efficiency of an optimized analysis by reducing

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the number of variables to be optimized for a given accuracy. The computation cost is about one-sixth of that of the optimized least-squares method for a given accuracy.

Least-Squares Rational-Function Approximation

The least-squares approximation uses a rational function, which is a second-order polynomial in the Laplace variable with an additional series of simple poles, for each term of the generalized unsteady aerodynamic matrix $[Q(s)]$. The poles, which denote lag terms in the time domain, are common for all the elements of $[Q(s)]$, thereby considerably reducing the number of augmented aerodynamic states compared to the case where all (or some) of the elements are allowed to have different poles. This leads to the following representation of the $[Q(s)]$ matrix:

$$[Q(s)] = q\left([A_0] + [A_1]s(b/U) + [A_2]s^2(b/U)^2 + (U/b)\sum_{n=1}^{n_L} \frac{[A_{(n+2)}]}{s + (U/b)b_n} \right) \tag{3}$$

where b_n are the lag parameters (or the poles) of $[Q]$, q the freestream dynamic pressure, b the reference length, and U the freestream velocity. The original approximation of Roger,[7] Abel,[8] and the recent extension by Tiffany and Adams[9] multiplies each lag term by the Laplace variable s. The present study has determined that there is no need to include the Laplace variable in the numerator of the lag terms. However, $[A_0]$ is no longer the static air-force matrix. The accuracy of the approximation crucially depends on the values of the lag parameters b_n.

The values for the lag parameters must be positive for the stability of the transfer function. The number of lag parameters taken directly influences the fit accuracy of the approximate aerodynamic transfer function with the frequency domain data because the lag terms account for the lag associated with circulation, which is presumably represented exactly only by an infinite number of lag terms. When the inverse Laplace transform is applied on $[Q(s)]$, the approximate aerodynamic unit impulse response matrix results. Upon taking the convolution integral of the latter with the transient structural motion $\{\xi(t)\}$, the approximate transient air-force vector becomes available in the time domain. Now, this vector is included in the equation of motion of the structure, Eq. (1):

$$[M]\{\ddot{\xi}(t)\} + [C]\{\dot{\xi}(t)\} + [K]\{\xi(t)\} = q[A_0]\{\xi(t)\} + q(b/U)[A_1]\{\dot{\xi}(t)\} + q(b/U)^2[A_2]\{\ddot{\xi}(t)\} + q\sum_{n=1}^{n_L}[A_{(n+2)}]\{\xi_a(t)\}_n \tag{4}$$

where the augmented states are defined by

$$\{\xi_a(t)\}_n = \int_0^t \{\xi(\tau)\} \exp[-(U/b)b_n(t-\tau)]\, \mathrm{d}\tau \tag{5}$$

or, equivalently, by the following differential equations:

$$\{\dot{\xi}_a(t)\}_n = \{\xi(t)\} - (U/b)b_n\{\xi_a(t)\}_n \tag{6}$$

The equivalent state-space formulation of Eqs. (4) and (6) is given by

$$\{\dot{X}\} = [A]\{X\} \tag{7}$$

where

$$\{X\} = [\{\dot{\xi}(t)\}^T, \{\xi(t)\}^T, \{\xi_a(t)\}_n^T]^T \tag{8}$$

$$[A] = \begin{bmatrix} -[\bar{M}]^{-1}[\bar{C}] & -[\bar{M}]^{-1}[\bar{K}] & [\bar{M}]^{-1}[\bar{A}_3] & \cdots & [\bar{M}]^{-1}[\bar{A}_{n_L+2}] \\ [I] & [0] & [0] & \cdots & [0] \\ [0] & [I] & -(U/b)b_3[I] & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & [0] \\ [0] & [I] & [0] & \cdots & -(U/b)b_{n_L}[I] \end{bmatrix} \tag{9}$$

and where

$$[\bar{M}] = [M] - q(b/U)^2[A_2] \tag{10a}$$

$$[\bar{K}] = [K] - q[A_0] \tag{10b}$$

$$[\bar{C}] = [C] - q(b/U)[A_1] \tag{10c}$$

$$[\bar{A}_{(n+2)}] = q[A_{(n+2)}] \tag{10d}$$

Notice that, if the number of retained modes is N and the number of lag states is n_L, then the total number of state-space equations is $2N + Nn_L$ and the size of the augmenting state vector is Nn_L, which implies that there are exactly N augmented states per lag parameter b_n.

The squared error between the approximate transfer-function matrix $[Q(s)]$ and the tabulated frequency domain data $[H(k)]$ at given values of reduced frequencies, $k = \omega b/U$, for which $sb/U = ik$ is given by

$$\epsilon_{ij} = \sum_{m=1}^{N_k} \frac{|Q_{ij}(ik_m) - H_{ij}(k_m)|^2}{M_{ij}(k_m)} \tag{11}$$

where N_k is the number of reduced frequencies at which fit is desired, and

$$M_{ij}(k_m) = \max\{1, |H_{ij}(k_m)|^2\}$$

The linear coefficients $(A_n)_{ij}$ are determined such that the squared error, given by Eq. (11), is the minimum. This can be translated into the following equation:

$$\frac{\partial \epsilon_{ij}}{\partial (A_n)_{ij}} = 0 \tag{12}$$

where $n = 0, \ldots, n_L + 2$. This yields

$$\{A_{ij}\} = ([F]^T[\bar{F}] + [\bar{F}]^T[F])^{-1}([\bar{F}]^T\{H_{ij}\} + [F]^T\{\bar{H}_{ij}\}) \tag{13}$$

where $\{H_{ij}\}$ is the vector of the evaluations of $H_{ij}(k)$ at the chosen frequencies, $\{A_{ij}\}$ the vector of the $n_L + 3$ elements, $(A_n)_{ij}$ the vector of the coefficient matrices, and the ith row of $[F]$ is

$$\{F_i\} = \left\{ 1 \quad i\omega_i \quad -\omega_i^2 \quad \frac{1}{i\omega_i + b_1} \cdots \frac{1}{i\omega_i + b_{n_L}} \right\} \tag{14}$$

The barred quantities $[\bar{F}]$ and $\{\bar{H}\}$ represent the complex conjugates of $[F]$ and $\{H\}$, respectively. The dimension of the least-squares fitting process can be reduced by imposing constraints on the elements of the transfer function to match the oscillatory data at some values of the reduced frequency. This degrades the fit at other frequencies. No constraints were imposed on the transfer function in the present work.

Optimized Least-Squares Approximation

As commented earlier, the nature of the least-squares fit of the approximate transfer function with frequency domain data is sensitive to the values of the lag parameters b_n in the rational form of the generalized air forces. In most of the formulations, these values have been chosen on the basis of experience with a given configuration and have been held constant over a range of Mach numbers. Only recently have there been attempts to include these nonlinear parameters as design variables and to perform a nonlinear optimization over them (cf. Refs. 9, 16, 17). Since the lag parameters appear in the denominator of the aerodynamic transfer function, when they are chosen as variables in the least-squared error curve-fitting process, the associated gradient conditions are no longer linear. Dunn[16] and Karpel[17] used gradient-based techniques to solve these equations. Whenever such a finite-difference technique is used, there are difficulties associated with the choice of increment in the design variables (in this case, the lag parameters) to compute the gradient, as well as with the choice of the differencing scheme, which are issues critical to stability and convergence. To avoid these problems and to accelerate convergence, Nelder and Mead[18] proposed a sequential simplex nongradient optimizing scheme. Its advantages over gradient-based techniques have been verified by several applications in various fields, and it has been used recently by Tiffany and Adams[9] in the field of aeroelasticity. The present study also employs the simplex optimizer, which is valuable in applications wherein the gradients of the objective function are not available in a closed form. The objective function for minimization in the nonlinear optimizer is defined as the weighted sum of the normalized least-squared errors of all of the elements of the unsteady transfer-function matrix at all the reduced frequencies desired for the fitting process. This objective function, therefore, is

$$J = \sum_{m=1}^{N_k} \sum_{i=1}^{N} \sum_{j=1}^{N} w_{ij} \epsilon_{ij} \tag{15}$$

The weighting factors w_{ij} are desired if some elements of the matrix are more important in fitting with the frequency domain data than others. However, in this study, all the elements of $[Q(s)]$ were considered equally important, and, therefore, the weighting factors are all taken to be unity. A similar form of the objective function is used by Tiffany and Adams.[9]

The process of nonlinear optimization from which the values of the lag parameters b_n are determined is essentially the same as that employed by Tiffany and Adams[9] in their extended least-squares method. The constraints on b_n are incorporated by using a wall-type penalty function in this work, which makes the objective function assume a large value whenever a constraint is violated. This is sufficient since the starting point for the process is always within the feasible space and, therefore, no discontinuities are introduced.

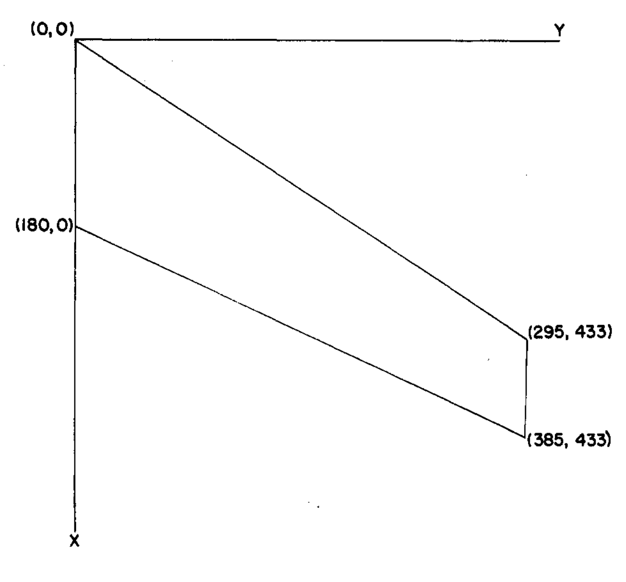

Fig. 1 Wing planform geometry for the test case.

Numerical Example

The optimized least-squares method is applied to a high aspect ratio wing, shown in Fig. 1. Only the first six structural modes are retained, without considering any control surface motions. The first six natural frequencies of the wing are shown in Table 1. To generate the frequency domain aerodynamics for this wing, the doublet-lattice method[19] is used. The reduced frequency, $k = \omega b/U$, is based on reference length, $b = 67.5$ in., which is the semi-root-chord of the test wing. The generalized air-force data are generated for 11 reduced frequencies: 0.0, 0.05, 0.1, 0.125, 0.175, 0.2, 0.3, 0.5, 0.7, 0.9, and 1.2.

More points are chosen at low values of k because a better curve fit at low reduced frequencies is considered more important in view of flutter. First, we demonstrate the advantage of using nonlinear optimization over the basic, unoptimized least-squares curve-fitting process. In the unoptimized least-squares method, the values of the nonlinear lag parameters b_n are chosen only once and remain fixed for all Mach numbers and reduced frequencies. The unoptimized lag-parameter values used in this investigation are given in Table 2. Figure 2 shows the fit for the (1,1) element of the generalized air-force matrix at $M = 0.9$; it is typical. Both the unoptimized and the optimized least-squares methods are shown. It is seen that, due to optimization, the number of augmented states in the resulting state-space equations is reduced, with no degradation in the quality of fit. The optimized least-squares method requires two lag states (amounting to 12 augmented states in the state-space matrix), whereas the unoptimized method requires four lag states (24 augmented states) for the same accuracy of fit with the frequency-domain data.

Phenomenon of Repeated Poles

Table 3 displays the optimum values for the lag parameters b_n in the least-squares rational approximation. These are seen to differ considerably from the baseline values seen in Table 2. One of the more interesting differences lies in the phenomenon of repeated poles. This phenomenon of pole multiplicity can be observed in Table 3 where, for several Mach numbers, two or more lag parameters have values quite close to one another. An interesting Mach number is 0.90 for which any approximation that has more than one lag state turns out to be the case of repeated poles. This repetition of the lag-parameter values has a special significance in the transient response analysis, which will be analyzed presently.

Table 1 Structural properties of the test wing

First six natural frequencies, Hz
1.74
4.94
9.31
13.2
21.1
26.0

Table 2 Lag parameters for unoptimized least-squares method

Number of lag states n_L	Values of lag parameters b_n
1	0.176
2	0.067, 0.317
3	0.027, 0.136, 0.464
4	0.010, 0.100, 0.300, 1.000

New Power Series Representation of the Lag States

As seen earlier, the optimization process in many cases leads to a solution with two or more lag parameters virtually equal. Upon a thorough investigation of the topography of the objective function J, it was found that in those cases J had absolute minima running near and along the line $b_1 = b_2$ when only two lags were close to each other, or the line $b_1 = b_2 = b_3$ when three lags were close to each other, and so on. Whenever any two, or more, of the lag parameters have values close to each other, the first matrix on the right side of Eq. (13) has two, or more, columns nearly identical, and this makes the matrix very close to being singular. Consequently, when the inverse of the matrix is taken to solve for the linear coefficients, the linear lag coefficients $[A_{(n+2)}]$ have elements corresponding to the lag parameters that are close, very large, of opposite sign, and differing in magnitude by small amounts. Such lag coefficients make the subsequent eigenvalue problem poorly conditioned. To avoid this, an alternative method is sought to approximate the transfer function when two or more of its lag parameters of the conventional type have optimum values close to each other. If we consider an approximation with only two lag states, then typical optimum lag terms for the repeated pole case can be combined in the following manner:

$$Ae^{-b_1t} - (A+\epsilon)e^{-(b_1+\delta)t} = A(e^{-b_1t} - e^{-b_1t}e^{-\delta t}) - \epsilon e^{-b_1t}e^{-\delta t} \qquad (16)$$

Here, the lag coefficients are $A_3 = A$ and $A_4 = -(A+\epsilon)$; the respective optimum lag parameters are b_1 and $(b_1+\delta)$, where ϵ and δ are small in magnitude. If a Taylor series expansion is used for the exponential $e^{-\delta t}$, then

$$e^{-\delta t} = 1 - \delta t + \tfrac{1}{2}(\delta t)^2 + \cdots \qquad (17)$$

Since δ is a small number, its powers that are greater than unity can be neglected and the combined lag terms become

$$A[e^{-b_1t} - (1-\delta t)e^{-b_1t}] - \epsilon(1-\delta t)e^{-b_1t} \simeq -A\epsilon e^{-b_1t} + A\delta te^{-b_1t} \qquad (18)$$

(The product $\epsilon\delta$ is small compared to 1.) The lag terms given by Eq. (18) point toward a new approximation which, converted to the Laplace domain, becomes

$$\frac{-A\epsilon}{s+b_1} + \frac{A\delta}{(s+b_1)^2}$$

This approximation was tried and it produced reasonable linear lag coefficients for the same value of total fit error as that of the original formulation. The same sort of power series approximation worked for the cases where three and four lag parameters of the original formulation were close to each other. In these cases, the highest order terms of the new power series were of third and fourth order, which implies that the

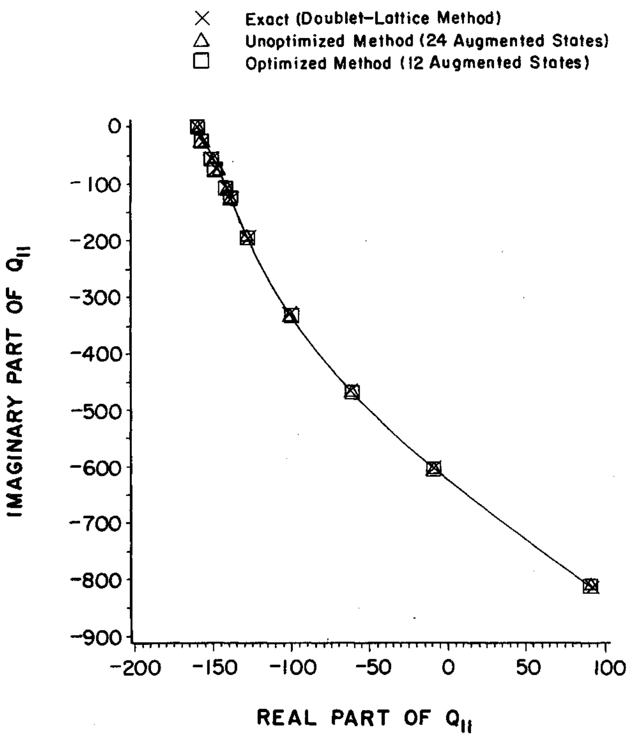

Fig. 2 Curve fits of element (1,1) of the generalized aerodynamics matrix at $M = 0.9$.

Table 4 New approximations for $M = 0.90$

Number of lag states	Lag parameters: Conventional	Lag parameters: New
2	0.218220 0.218540 (fit error = 4.197)	0.2185 (double pole) (fit error = 4.1967)
3	0.142086 0.142831 0.142031 (fit error = 3.259)	0.1425 (triple pole) (fit error = 3.257)
4	0.172794 0.17281 0.17733 0.603264 (fit error = 0.6112)	0.175295 (triple pole) 0.58698 (simple pole) (fit error = 0.6107)

Table 3 Optimum lag-parameter values

	Lag-parameter values			
Mach number M	b_1	b_1, b_2	b_1, b_2 b_3	b_1, b_2 b_3, b_4
0.2	0.24525	0.25100, 2.00000	0.22813, 2.99994, 1.07957	0.13306, 0.31917, 2.00959, 2.02007
0.4	0.23725	0.25206, 1.03253	0.23103, 1.24763, 1.24600	2.00000, 0.20533, 1.54040, 0.98936
0.6	0.26900	1.62539, 0.26082	3.51629, 3.49729 0.22940	0.20346, 1.39337, 1.35035, 1.39999
0.7	0.29455	0.30719, 1.11239	0.11778, 0.11780, 0.20877	1.83290, 1.72690, 1.81580, 0.20360
0.8	0.36725	0.04616, 0.38490	0.00064, 0.45892, 0.63202	0.36926, 0.36497, 0.37030, 0.33444
0.9	0.32100	0.21822, 0.21854	0.14209, 0.14283, 0.14203	0.17279, 0.17281, 0.17733, 0.60326
0.95	0.20332	0.15574, 0.41903	0.17579, 1.39223, 0.22497	0.06481, 0.23659, 0.23659, 1.49611

coefficients of these terms in the time domain must contain second- and third-order time dependence, respectively. The interesting case of $M = 0.9$ in the numerical example is examined in Table 4, where each of the repeated pole cases is replaced by a new approximation of multiple poles and the respective fit errors are examined. The errors of the conventional and new approximations are very nearly the same, but whereas one produces a poorly conditioned eigenvalue problem in the transient response analysis, the other does not. These observations lead one to believe that in many cases the simple pole series approximation for the lag terms is not appropriate and a series of higher order poles yields an equivalent fit, but with a meaningful result for the lag coefficients. (Such a pole series was also suggested by Richardson,[12] but for different reasons.) Therefore, for many cases

$$[Q(s)] = q\Bigg([A_0] + [A_1]s(b/U) + [A_2]s^2(b/U)^2 + (U/b)\sum_{n=1}^{N_1}\frac{[A_{(n+2)}]}{s+(U/b)b_n} + (U/b)^2\sum_{n=N_1+1}^{N_2}\frac{[A_{(n+2)}]}{[s+(U/b)b_n]^2} + \cdots\Bigg) \tag{19}$$

is a more useful approximation than the conventional approximation of Eq. (3). Here, N_1 is the total number of poles, $(N_2 - N_1)$ the number of poles repeated two or more times, and so on. However, there is another reason for preferring the new scheme to the conventional approximation. The new approximation has exactly the same number of augmented states as the conventional one for the same fit accuracy but requires drastically less computation time in the nonlinear optimizing process because the dimension of the simplex is reduced several times. For a second-order pole of the new form, which is equivalent to two conventional lag states, the simplex has to optimize for only one variable. For a fourth-order pole of the new type, the simplex still has one nonlinear variable, whereas the conventional method would require four nonlinear variables for the same fit accuracy. This reduces the computation time for optimization significantly. Even when the lags are of a mixed type (like one lag with multiplicity 2 and two lags with multiplicity 1), the reduction in computational time is remarkable. The inverse Laplace transform of Eq. (19) yields the following:

$$[h(t)] = q\{[A_0]\delta(t) + (b/U)[A_1]\dot{\delta}(t) + (b/U)^2[A_2]\ddot{\delta}(t) + (U/b)\sum_{n=1}^{N_1}[A_{(n+2)}]\exp[-(U/b)b_n t] + (U/b)^2\sum_{n=N_1+1}^{N_2}[A_{(n+2)}]t\exp[-(U/b)b_n t]\cdots\} \tag{20}$$

From Eq. (20), it can be seen that the new series for lag terms will have the same number of augmented states in the state-space equations (however, the equations themselves will now be different) for the same fit accuracy as the ill-conditioned conventional approximation. It can be observed that now the decaying exponentials are time weighted. The equation of motion for the new approximation of lag states becomes the following:

$$[\bar{M}]\{\ddot{\xi}(t)\} + [\bar{C}]\{\dot{\xi}(t)\} + [\bar{K}]\{\xi(t)\} - (U/b)\sum_{n=1}^{N_1}q[A_{(n+2)}]\{\xi_a(t)\}_n - (U/b)^2\sum_{n=N_1+1}^{N_2}q[A_{(n+2)}]\{\xi_a(t)\}_n - (U/b)^3\sum_{n=N_2+1}^{N_3}q[A_{(n+2)}]\{\xi_a(t)\}_n - \cdots = \{0\} \tag{21}$$

Here, the state-space equations for the augmented states are as follows:

$$\{\dot{\xi}_a(t)\}_{n_1} = \{\xi(t)\} - b_n(U/b)\{\xi_a(t)\}_{n_1} \tag{22a}$$

$$\{\dot{\xi}_a(t)\}_{n_2} = \{\xi_a(t)\}_{n_1} - b_n(U/b)\{\xi_a(t)\}_{n_2} \tag{22b}$$

$$\{\dot{\xi}_a(t)\}_{n_3} = \{\xi_a(t)\}_{n_2} - b_n(U/b)\{\xi_a(t)\}_{n_3} \tag{22c}$$

etc., where the subscripts on the augmented states ξa associated with the various multiplicities are n_1, n_2, n_3, etc. The subscript n_1 varies from 1 to N_1, n_2 varies from $(N_1 + 1)$ to N_2, and n_3 varies from $(N_2 + 1)$ to n_3, and so on. From Eqs. (21) and (22) follow the state-space equations for the entire system. These can be written as

$$\{\dot{X}\} = [A]\{X\} \tag{23}$$

$$\{X\} = \begin{bmatrix} \{\dot{\xi}(t)\} \\ \{\xi(t)\} \\ \{\xi_a(t)\}_{n_1} \\ \{\xi_a(t)\}_{n_2} \\ \{\xi_a(t)\}_{n_3} \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} \tag{24}$$

$$[A] = \begin{bmatrix} -[\bar{M}]^{-1}[\bar{C}] & -[\bar{M}]^{-1}[\bar{K}] & q[\bar{M}]^{-1}[\bar{A}_1] & \cdots & q[\bar{M}]^{-1}[\bar{A}_{N_3}] \\ [I] & [0] & [0] & [0] & [0] \\ [0] & [I] & [R_1] & [0] & [0] \\ [0] & [0] & [I] & [R_2] & [0] \\ [0] & [0] & [0] & [I] & [R_3] \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{25}$$

where $[\bar{M}]$, $[\bar{C}]$, and $[\bar{K}]$ are the same as those defined in Eqs. (10),

$$[R_1] = -(U/b)\begin{bmatrix} b_1[I] & & 0 \\ & \ddots & \\ 0 & & b_{N_1}[I] \end{bmatrix} \tag{26a}$$

$$[R_2] = -(U/b)\begin{bmatrix} b_{(N_1+1)}[I] & & 0 \\ & \ddots & \\ 0 & & b_{N_2}[I] \end{bmatrix} \tag{26b}$$

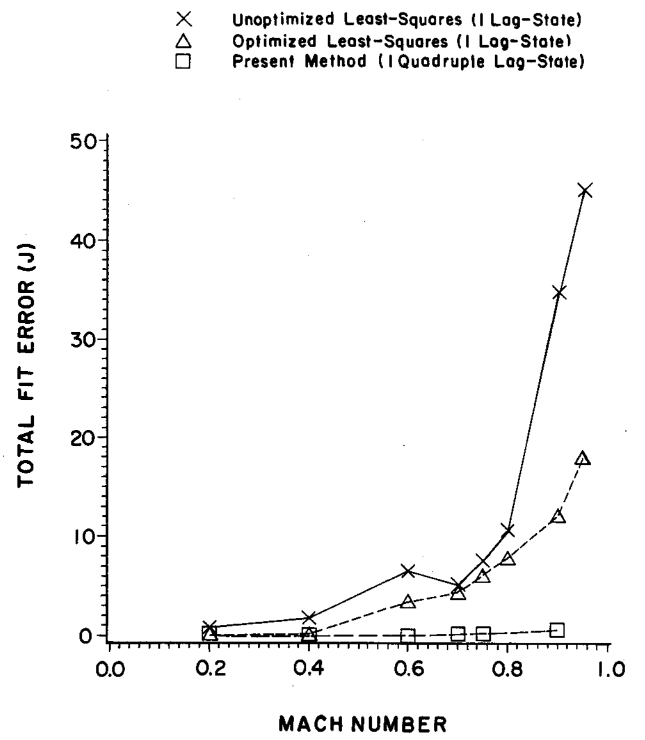

Fig. 3 Variation of the total fit error with Mach number with one lag state in the conventional method and one quadruple lag state in the present method.

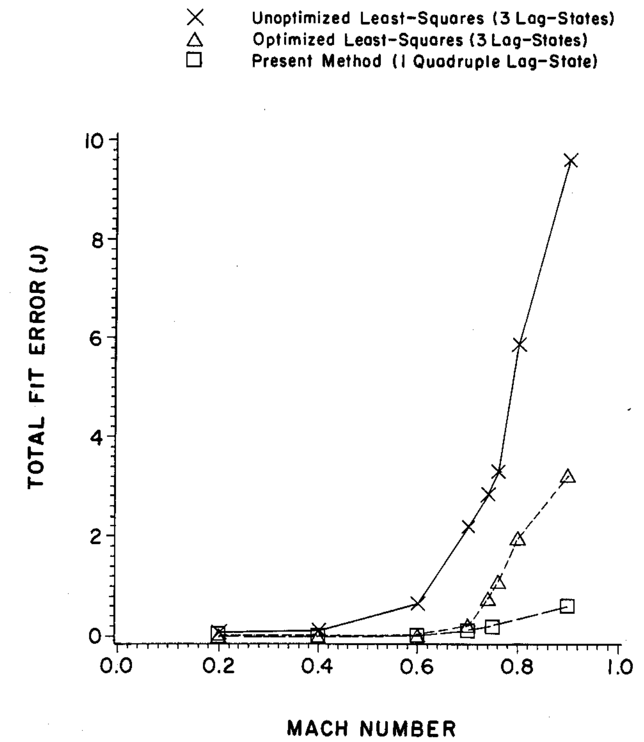

Fig. 5 Variation of the total fit error with Mach number with three lag states in the conventional method and one quadruple lag state in the present method.

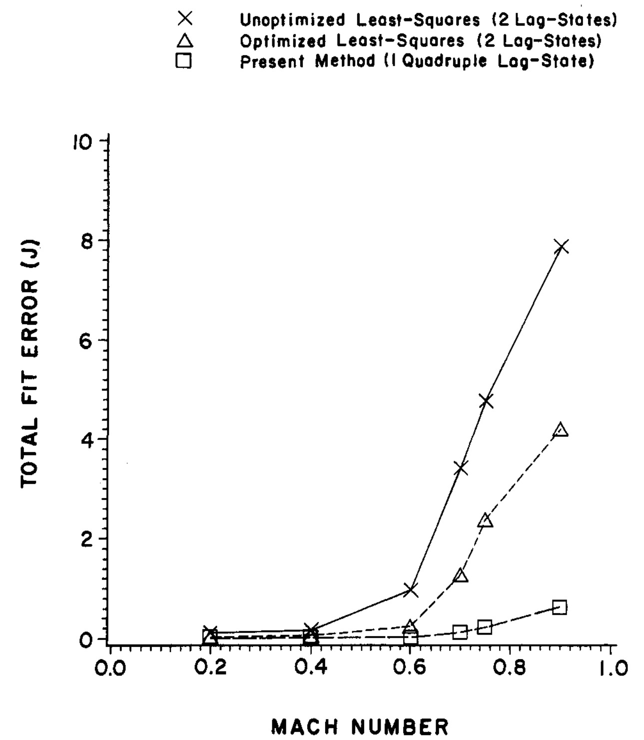

Fig. 4 Variation of the total fit error with Mach number with two lag states in the conventional method and one quadruple lag state in the present method.

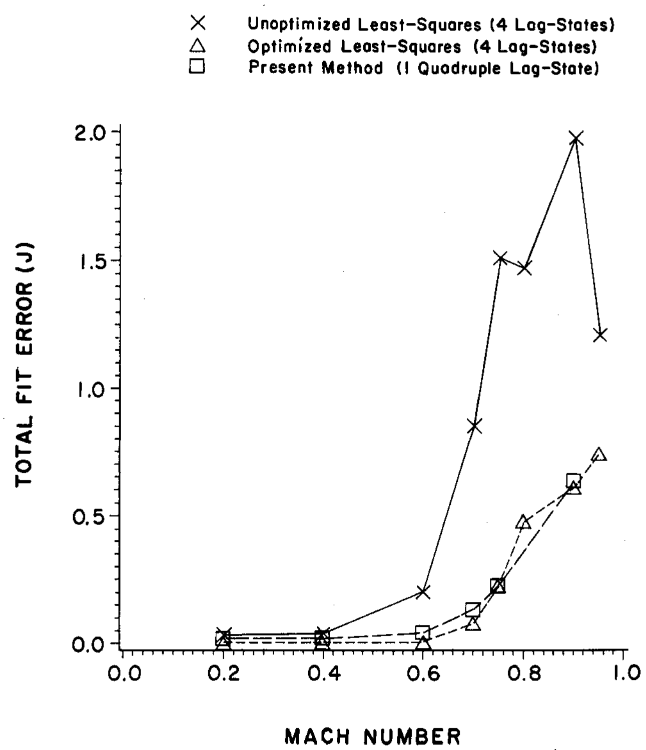

Fig. 6 Variation of the total fit error with Mach number with four lag states in the conventional method and one quadruple lag state in the present method.

$$[R_3] = -(U/b)\begin{bmatrix} b_{(N_2+1)}[I] & & 0 \\ & \ddots & \\ 0 & & b_{N_3}[I] \end{bmatrix} \tag{26c}$$

etc., and $[I]$ is the identity matrix of the order N, which is the number of structural degrees of freedom. From the state matrix defined by Eq. (21), it is evident that the total number of state-space equations in the new formulation is $2N + NN_m$, where N_m is the total number of poles, i.e., the number of distinct poles added to the number of poles repeated only twice, which in turn is added to the number of poles repeated only three times. This process continues until the number of poles repeated the largest number of times (m times) has been added to the sum. Since N_m is nothing but n_L of the conventional approximation, there is no increase in the total number of state-space equations in the multiple pole approximation.

Efficient Approximation Using Multiple Poles

For a given number of augmented states, a multiple pole approximation is more efficient in the optimization process because of the reduced dimension of the simplex. A logical result of this observation is the use of the multiple pole approximation for all cases. In order to test this, we use

$$[Q(s)] = q\Bigg([A_0] + [A_1]s(b/U) + [A_2]s^2(b/U)^2 + \sum_{n=1}^{4}(U/b)^n \frac{[A_{n+2}]}{[s + (U/b)b_1]^n}\Bigg) \tag{27}$$

Although this approximation arose strictly for that case in the modified optimization where four lag parameters had similar values, it will be used for further comparisons with the unoptimized and the conventionally optimized least-squares methods. A detailed analysis of the fit accuracies of the different methods is available in Figs. 3–6. Figure 3 compares the normalized least-squares error function J for the three methods when only one lag state is incorporated in the rational function of the conventional approximation. As seen in the figure, the conventionally optimized fit error is appreciably less than the baseline error for all Mach numbers considered. The unoptimized error is at least twice as large as the conventionally optimized error and rises more steeply at higher Mach numbers than does the latter. The present method is seen to be several times more accurate than either of the least-squares methods. Figure 4, for 2 lag states in the conventional approximation, shows that the initial trend of fit improvement due to optimization is preserved and the present method is still the most accurate. Figure 5 presents the same information for three traditional lag states compared with the quadruple pole of the present method. Here, the fit improvement due to optimization is greater, and the worsening of fit with increase of Mach number is more gradual for the optimized method than for the unoptimized one. Figure 6, for four lag states, reveals an interesting feature that was not observed previously. The unoptimized fit for $M = 0.8$ and 0.9 is worse than for $M = 0.95$. Also, when Figs. 4 and 5 are compared, it is seen that the unoptimized least-squares method gives a greater fit error with three lag states than with two lag states. These are some of the hazards associated with keeping the values of lag parameters the same for all Mach numbers. The optimized fit error is more consistent with Mach number and is substantially less than the unoptimized one. Figure 6 shows that the present method has a fit error similar to the conventionally optimized method with four lag parameters.

When Table 3 is seen in the context of Fig. 6, an important fact is observed. Table 3 shows that all of the four lag parameters in the conventionally optimized method are close to one another only at $M = 0.9$, but not at other values of the Mach number. However, it can be seen in Fig. 6 that even at those Mach number values at which the conventional approximation produces nonrepetitive pole values, the new approximation of quadruple pole is of similar fit accuracy to the conventionally optimized method. This leads to the conclusion that a multiple pole approximation can be used even in those cases where the conventional approximation does not call for that number of repeated poles, without any appreciable loss in accuracy. This is a distinct advantage in terms of computational efficiency, as will be illustrated later.

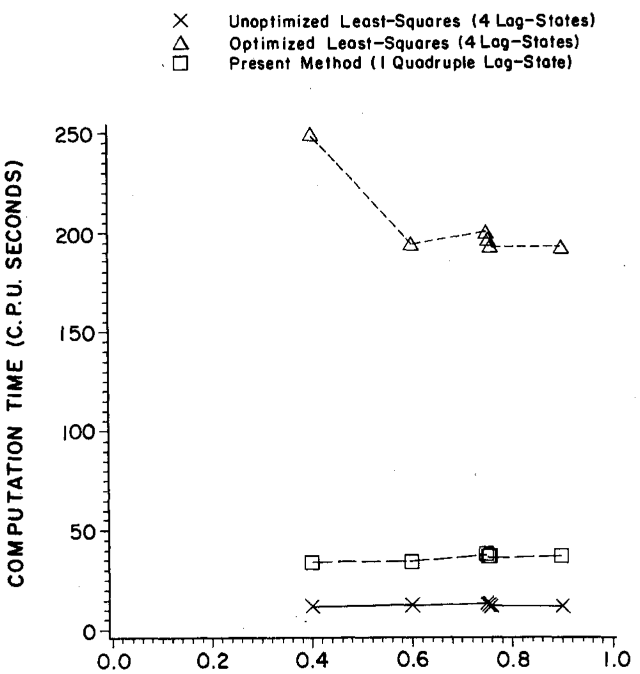

Fig. 7 Variation of total computation time with Mach number for the three approximations.

Comparison of Computational Costs

Since the present method uses a quadruple pole, the simplex procedure has only one variable to optimize. The conventional method has four optimization variables for a similar fit accuracy (Fig. 6). As the computation cost in the optimization process increases with the number of variables, the present method is expected to be more efficient in the curve-fitting process. In the solution of the eigenvalue problem, the computation time increases with the number of augmented states in the state-space matrix. The total computation time is the sum of the curve-fit time and the eigenvalue analysis time. At $M = 0.75$, comparisons between the conventional and present methods yield the following data. The conventionally optimized method requires a total computation time of 200.3 CPU s on an IBM 4381. The present method with a quadruple pole requires only 38.3 CPU s. Both the methods have a fit-error objective function of 0.22. With three lag states, the conventional method requires 126.7 CPU s; with two lag-states, 33.5 CPU s; and with 1 lag state, 15.5 CPU s. For any number of conventional lag states less than four, the present method is many times more accurate (Figs. 3–5) in terms of the total fit error than the conventional method. A comparison between the total CPU time of the conventional optimized and unoptimized methods with the present method is shown in Fig. 7 for different Mach numbers. The figure shows that the total CPU time of the present method is about 1/5–1/6 of that of the conventionally optimized method. The present method requires a CPU time about 3 times that of the unoptimized method. The relative fit accuracies of these methods were seen in Fig. 6. These comparisons indicate that the present method gives a similar fit accuracy with only a small fraction of the total computation cost of the conventionally optimized least-squares method.

Conclusions

When repeated poles occur in the rational polynomial approximation of the unsteady aerodynamic transfer function, the previous methods of approximation lead to a poorly conditioned eigenvalue problem. The latter is avoided in the present method through the use of a multiple pole series approximation for the lag states. Although arising out of the phenomenon of repeated poles in the conventional method, the new multiple pole approximation can be applied for even those Mach numbers for which repeated poles do not occur in the conventional approximation. This results in a drastic reduction in the total computation cost of the optimized eigenvalue analysis, without causing any significant loss in accuracy. The present method, therefore, makes it possible to carry out optimized transient response analysis at a reduced cost while avoiding the ill-conditioned eigenvalue problem of previous procedures.

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